ISRAEL JOURNAL OF MATHEMATICS, Vol. 32, No. 4, 1979

CONTINUOUS MAPS OF THE CIRCLE WITHOUT PERIODIC POINTS

BY

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ABSTRACT

This paper gives a description of all the continuous maps of the circle without periodic point.

The homeomorphisms of the circle without periodic points have been completely classified. If the homeomorphism is minimal (every orbit is dense) then it is conjugate to a rotation through an irrational angle. Otherwise, there is a unique minimal set M which is nowhere dense in the circle $\mathbf{T}$. The complementary intervals are permuted freely (without finite cycles) among themselves by the homeomorphism [1].

Starting with the latter class of homeomorphisms, it is easy to obtain continuous (non one to one) maps of $\mathbf{T}$ without periodic points. In fact, if $f' : \mathbf{T} \to \mathbf{T}$ is any continuous map which agrees with the given homeomorphism f on M and such that $f'(I) = f(I)$ for any complementary interval I, then (since $f^k(I) \cap I = \emptyset$ for $k \geqq 1$) certainly f' has no periodic points. Further we can collapse any positively invariant set of intervals $\{I_j\}_{j \in \mathcal{J}}$ to points p_j and, if this set of intervals is not negatively invariant, the induced map is continuous, not one to one, and has no periodic points.

The purpose of this note is to show that all continuous maps of the circle without periodic points arise in this manner. In particular, as we shall see, if f is not a homeomorphism, it cannot be minimal, or even have a dense orbit.

Our standing assumption is that f is a continuous map of the circle $\mathbf{T}$ to itself, without periodic points (that is, if $x \in \mathbf{T}$ and $j > 0$, then $f^j(x) \neq x$). Our primary interest is in the case that f is not a homeomorphism. However, except when hypotheses exclude them, our results hold for homeomorphisms as well.

If $x \in \mathbf{T}$ the forward orbit (or just orbit) of x is the set $\mathcal{O}(x) =$

Received May 4, 1978

$[f^j(y) \mid j = 0, 1, 2, \cdots]$, and the omega limit set of x, $\Omega(x) = \bigcap_{m \geqq 0}$ closure $\mathcal{O}(f^m(x))$; $\Omega(x)$ is the set of $y \in \mathbf{T}$ for which there is a sequence $n_i \to \infty$ such that $f^{n_i}(x) \to y$.

We note that f must be onto (if not, $f(\mathbf{T}) = I$ is homeomorphic to a closed real interval and $f(I) \subset I$, so f has a fixed point).

The following lemma is the key to the analysis of the non one to one case.

LEMMA 1. *Suppose there are distinct points y_1 and y_2 in $\mathbf{T}$ with $f(y_1) = f(y_2) = z$. Let I be the arc in $\mathbf{T}$ with endpoints y_1 and y_2 such that $z \not\in I$. Then the orbit of z never enters I.*

PROOF. If the conclusion is false, there is a $m > 1$ such that $f^m(y_1) = f^m(y_2) \in I$. Lift f to a map $\tilde{f} : \mathbf{R} \to \mathbf{R}$ with $x < \tilde{f}(x) < x + 1$. The assumption that f has no fixed points implies that $\tilde{f}^k(x+1) = \tilde{f}^k(x) + 1$ $(k = 1, 2, \cdots)$.

We may suppose that $0, a, b, \in \mathbf{R}$ correspond, respectively, to y_1, y_2, and $z \in \mathbf{T}$, and $\tilde{f}(0) = \tilde{f}(a) = b > a$. (If $\tilde{f}(a) = \tilde{f}(0) + k$, where k is a non-zero integer, then the graph of $y = \tilde{f}(x)$ would cross the line $y = x$ or $y = x + 1$, and f would have a fixed point on $\mathbf{T}$.)

Let r be a positive integer such that $r < \tilde{f}^m(0) = \tilde{f}^m(a) < r + 1$. Since $f^m(y_1) = f^m(y_2) \in I$ we have that $r < \tilde{f}^m(0) = \tilde{f}^m(a) < r + a$. But then the graph of $y = \tilde{f}^m(x)$ must cross the line $y = x + r$ at some point x'. That is, there is an $x' \in [0, a]$ such that $\tilde{f}^m(x') = x' + r$. If y' denotes the corresponding point of $\mathbf{T}$, then $f^m(y') = y'$ and y' is a periodic point of f, contrary to our standing hypothesis.

COROLLARY 1. *If $f : \mathbf{T} \to \mathbf{T}$ is minimal then f is a homeomorphism.*

Let a, b, and c be distinct points of $\mathbf{T}$. We say that b is between a and c if b is on the counterclockwise arc from a to c.

LEMMA 2. *All orbits of f have the same order type. That is, if $x, x' \in \mathbf{T}$ and $f^m(x)$ is between $f^r(x)$ and $f^k(x)$, then $f^m(x')$ is between $f^r(x')$ and $f^k(x')$.*

PROOF. We may suppose that $r = 0$. If $a, b \in \mathbf{T}$ with $a \neq b$, let $l_{(a,b)}$ be the length of the counterclockwise arc from a to b. Let m and k be positive integers. For $z \in \mathbf{T}$, let $\pi(z) = l_{(z, f^k(z))} - l_{(z, f^m(z))}$. Since f has no periodic points, π is continuous and never zero. Moreover, $f^m(x)$ is between x and $f^k(x)$ if and only if $\pi(x) > 0$. Thus, if $x, x' \in \mathbf{T}$, $\pi(x) > 0$ if and only if $\pi(x') > 0$. This completes the proof.

LEMMA 3. *Let $x \in \mathbf{T}$, let k and l be non-negative integers, and let I be an arc on $\mathbf{T}$ with endpoints $f^k(x)$ and $f^l(x)$. Then there is an $m > 0$ with $f^m(x) \in I$.*

PROOF. We may suppose $l = 0$. Suppose the orbit of x never enters I. Then (say) $f^k(x)$ is between x and $f^{2k}(x)$, and $f^{2k}(x) \not\in I$. By Lemma 2 (with $x' = f^k(x)$), $f^{2k}(x)$ is between $f^k(x)$ and $f^{3k}(x)$, and $f^{3k}(x) \not\in I$. Since $f^{nk}(x) \not\in I$, we have $f^{nk}(x)$ is between $f^{(n-1)k}(x)$ and x, and it follows that $\lim_{n\to\infty} f^{nk}(x) = x'$ exists. Thus $f^k(x') = x'$ and we have a periodic point.

We remark that one can prove the existence of the rotation number α as well as Poincaré's theorem (all orbits have the same order type as rotation by α); the proofs are word for word the same as for homeomorphisms ([1]). However, Lemmas 2 and 3 are sufficient for our purposes. A proof of Poincaré's theorem can be obtained from our main result.

For $x \in \mathbf{T}$, let J_x be the largest interval containing x such that $f^m(x) \not\in J_x$, for $m \geqq 1$. If ζ and ζ' are the endpoints of J_x, Lemma 3 tells us that $\zeta, \zeta' \neq f^k(x)$ for $k > 0$, and also that ζ and ζ' are in $\Omega(x)$.

If $x \in \Omega(x)$ then $x = \zeta$ or ζ' (or both, in which case $J_x = \{x\}$). However, if ζ or ζ' is different from x, then $\zeta \neq \zeta'$. For, if $\zeta = \zeta' \neq x$ then $\Omega(x) = \{\zeta\}$, and it follows immediately that $f(\zeta) = \zeta$.

Since the endpoints ζ, ζ' are in $\Omega(x)$, J_x can also be characterized as the largest interval containing x for which $f^m(x) \not\in J_x$, for $m \geqq m_0$ (for every positive integer m_0).

We also note that J_x is closed. (If not, there would be a "closest" $f^m(x)$ to x ($m \geqq 1$), again contradicting Lemma 3.)

In the following theorem, we list the properties of the sets J_x which will be of interest to us.

THEOREM 1. (i) *$f^m(x) \not\in f(J_x)$, for $m > 1$.*

(ii) *If ζ and ζ' are the endpoints of J_x, then $f(\zeta)$ and $f(\zeta')$ are the endpoints of $f(J_x)$.*

(iii) $f(J_x) \cap J_x = \varnothing$.

(iv) $f^m(J_x) = J_{f^m(x)}$ $(m = 1, 2, \cdots)$.

(v) *The intervals $J_{f^m(x)}$ $(m = 0, 1, 2, \cdots)$ are pairwise disjoint.*

(vi) *If $f(x) = f(x')$, then $J_x = J_{x'}$.*

(vii) *The sets $\{J_x\}$ form a decomposition of $\mathbf{T}$. (That is, if $x, y \in \mathbf{T}$ then $J_x = J_y$ or $J_x \cap J_y = \varnothing$).*

(viii) *At most countably many of the sets J_x are non-degenerate ($J_x \neq \{x\}$).*

(ix) *$J_x = J_y$ if and only if $J_{f^m(x)} = J_{f^m(y)}$ for some (every) $m \geqq 0$.*

PROOF. (i) If $m > 1$ such that $f^m(x) \in f(J_x)$, then, since $m - 1 > 0$, $f^{m-1}(x) \not\in J_x$. Then $f^m(x) = f(z)$, for some $z \in J_x$. That is, $f(f^{m-1}(x)) = f(z)$. By Lemma 1, there is an interval L with endpoints z and $f^{m-1}(x)$ such that

$f^k(x) \notin L$ $(k \geqq m)$. Thus the orbit of $f^m(x)$ never enters $L \cup J_x$. Since $z \in L \cap J_x$, $L \cup J_x$ is an interval and since $f^{m-1}(x) \notin J_x$, it is a strictly larger interval than J_x. This contradicts the maximality of J_x.

(ii) If, say, $f(\zeta) \in f(J_x)^\circ$ (the interior of $f(J_x)$) then, since $\zeta \in \Omega(x)$ $f^{m_i}(x) \to \zeta$ (when $m_1 \to \infty$) so $f^{m_{i+1}}(x) \in f(J_x)$, which contradicts (i).

(iii) If $f(J_x)$ consists of a single point (necessarily $f(x)$), then certainly $f(J_x) \cap J_x = \emptyset$. Otherwise $f(J_x)$ is a non-degenerate interval, and if $f(J_x) \cap J_x \neq \emptyset$, $f(J_x) \cup J_x$ is an interval containing one of the endpoints of J_x, say ζ, in its interior. Then $f^{m_i}(x) \to \zeta$ (where $m_i \to \infty$), so $f^{m_i}(x) \in f(J_x)$, contradicting (i) again.

(iv) $f^m(f(x)) = f^{m+1}(x) \notin f(J_x)$ for $m > 0$, by (i) so $J_{f(x)} \supset f(J_x)$. Since the endpoints ζ and ζ' of J_x are in $\Omega(x)$, the endpoints $f(\zeta)$ and $f(\zeta')$ of $f(J_x)$ are in $\Omega(f(x))$, and so $f(J_x)$ is the maximal interval containing $f(x)$ which $f^m(x)$ $(m > 1)$ does not enter. That is, $f(J_x) = J_{f(x)}$, and $f^m(J_x) = J_{f^m(x)}$ follows immediately by induction.

(v) We first show $J_{f^m(x)} \cap J_x = \emptyset$ $(m = 1, 2, \cdots)$. As in (iii) if $J_{f^m(x)} = \{f^m(x)\}$ the result is obvious. If not, and if $J_{f^m(x)} \cap J_x \neq \emptyset$, then $J_{f^m(x)} \cup J_x$ is an interval containing an endpoint ζ of J_x in its interior, and $f^{m_i}(x) \to \zeta$, where $m_i \to \infty$. Since $m_i > m$ for i large, $m_i = m + j_i$, where $j_i > 0$, we have $f^{j_i}(f^m(x)) \in J_{f^m(x)}$, which contradicts the definition of $J_{f^m(x)}$. Now, if $k > l > 0$, $k = l + r$, $J_{f^k(x)} \cap J_{f^l(x)} = J_{f^r(f^l(x))} \cap J_{f^l(x)} = \emptyset$.

(vi) By Lemma 1, there is an interval I with endpoints x and x' such that $f^m(x) = f^m(x') \notin I$ $(m \geqq 1)$, hence $I \subset J_{x'}$ by maximality of J_x.

Now $f^m(x') = f^m(x) \in J_x$ $(m \geqq 1)$, and $x' \in I \subset J_x$, so, by maximality of $J_{x'}$, $J_x \subset J_{x'}$. Interchanging x and x' we obtain $J_{x'} \subset J_x$.

(vii) If $y \in J_x$, $f^k(y) \in f^k(J_x)$ and $f^k(J_x) \cap J_x = \emptyset$, if $k > 0$, so $f^k(y) \notin J_x$. By maximality of J_y, $J_x \subset J_y$. Now $x \in J_x \subset J_y$, so $J_y \subset J_x$ and $J_x = J_y$.

(viii) Follows immediately from (vii).

(ix) If $J_x = J_y$ and $m \geqq 0$, then $J_{f^m(x)} = f^m(J_x) = f^m(J_x) = f^m(J_y) = J_{f^m(y)}$. Suppose $J_{f(x)} = J_{f(y)}$. Let J_x have endpoints ζ_1 and ζ_2, and J_y have endpoints r_1 and r_2. By (ii) f maps endpoints of J_x to endpoints of $f(J_x) = J_{f(x)}$ so $f(r_i) = f(\zeta_j)$, for some i, j. By (vi) $J_{r_i} = J_{\zeta_j}$. But $r_i \in J_y$, $\zeta_i \in J_x$ so $J_x = J_{\zeta_j} = J_{r_i} = J_y$ by (vii). If $J_{f^m(x)} = J_{f^m(y)}$ for $m > 1$, then $J_{f^{m-1}(x)} = J_{f^{m-1}(y)}$ and, by iteration, $J_x = J_y$.

From general considerations there is at least one minimal set for f. In fact, the minimal set is unique, as we now show. First we have a lemma, and a generalization of Corollary 1.

LEMMA 4. *Let $x, y \in \mathbf{T}$. Then $y \in \Omega(x)$ if and only if y is an endpoint of J_y.*

PROOF. Since the sets $J_{f^n(x)}$ are pairwise disjoint and $f(J_{f^n(x)}) = J_{f^{n+1}(x)}$ it is clear that no interior point of J_y can be in any $\Omega(x)$. Suppose, now, that $y \notin \Omega(x)$. Since $\Omega(x)$ is closed, there is an open arc V containing y with $V \cap \Omega(x) = \emptyset$. Choose V maximal, so the endpoints x_1 and x_2 of V are in $\Omega(x)$. Since $\Omega(x)$ is invariant $f^k(x_1) \notin V$ $(k \geqq 0)$, and $V \subset J_{x_1}$. Now $y \in V \subset J_{x_1}$, so $J_y = J_{x_1}$. Since V is open, $y \in J^\circ_{x_1} = J^\circ_y$.

COROLLARY 2. *If $f: \mathbf{T} \to \mathbf{T}$ has a dense orbit, then f is a homeomorphism.*

PROOF. If the orbit $\mathcal{O}(x)$ is dense in $\mathbf{T}$, then $\Omega(x) = \mathbf{T}$. By Lemma 4, all $y \in \mathbf{T}$ are endpoints of J_y. It follows that $J_y = \{y\}$, for all $y \in \mathbf{T}$. By Theorem 1, (vi), f must be a homeomorphism.

In the following theorem, $\mathcal{O}^-(x)$ is the "backward orbit" of x, $\mathcal{O}^-(x) = \bigcup_{k \leqq 0} f^k(x)$, and absolute value denotes cardinal number.

THEOREM 2. *There is exactly one minimal set M in $\mathbf{T}$, which consists of the endpoints of the intervals J_y $(y \in \mathbf{T})$. If x_1 and x_2 are endpoints of a non-degenerate interval $J_{x_1} = J_{x_2}$, then x_1 and x_2 are asymptotic $(d(f^n(x_1), f^n(x_2)) \to 0$, as $n \to \infty)$, and, if $y \in J^\circ_{x_1} = J^\circ_{x_2}$, then y is asymptotic to x_1 and x_2. If $x \in M$, then $|f^{-1}(x) \cap M| = 1$ or 2. There are at most countably many $x \in M$ for which $|f^{-1}(x) \cap M| = 2$ and if $x \in M$, there is at most one $x' \in \mathcal{O}(x) \cup (\mathcal{O}^-(x) \cap M)$ for which $|f^{-1}(x') \cap M| = 2$. If f is not a homeomorphism, then M is nowhere dense in $\mathbf{T}$.*

PROOF. The uniqueness of M and its characterization as the set of endpoints follows immediately from Lemma 4. Since the intervals $J_{f^n(x)} = f^n(J_x)$ are pairwise disjoint, their lengths tend to zero as $n \to \infty$, and the asymptotic assertions also follow.

Suppose now that x_1, x_2, x_3 are distinct points with $f(x_1) = f(x_2) = f(x_3) = x \in M$. We may suppose there is an interval I with endpoints x_1 and x_3 containing x_2 in its interior, and not containing x. By Lemma 1, $f^k(x_i) \notin I$ $(k \geqq 1)$, so $x_2 \notin \Omega(x_2)$, and therefore $x_2 \notin M$. Let $x \in M$ with $|f^{-1}(x) \cap M| = 2$, $f^{-1}(x) \cap M = \{z_1, z_2\}$. Since $f(z_1) = f(z_2)$, $J_{z_1} = J_{z_2}$ so z_1 and z_2 are endpoints of $J_{z_1} = J_{z_2}$. Since there are at most countably many non-degenerate intervals, the set of $x \in M$ with $|f^{-1}(x) \cap M| = 2$ is at most countable. Also $x = f(z_i)$ is the unique endpoint of $f(J_{z_i}) = J_x$ so $J_x = \{x\}$. That is, if $|f^{-1}(x) \cap M| = 2$, $J_x = \{x\}$ and if $z_1, z_2 \in M$ with $f(z_i) = x$, then $J_{z_1} = J_{z_2}$ is non-degenerate. Since $J_{z_i} \neq \{z_i\}$ $(i = 1, 2)$, this implies that $|f^{-1}(z_i) \cap M| = 1$. It follows by an easy induction that if $z \in \mathcal{O}^-(x) \cap M$, $|f^{-1}(z) \cap M| = 1$. If there were an $x' \in M$ with $x' \neq x$ and

$f(x') = f(x)$, then $J_x = J_{x'}$ and $x' \in J_x$ contradicting $J_x = \{x\}$. Thus $|f^{-1}(f(x)) \cap M| = 1$, and by induction $|f^{-1}(f^n(x)) \cap M| = 1$ $(n \geqq 1)$.

Suppose that f is not a homeomorphism. To show that M is nowhere dense, it is sufficient to show that there is no interval in $\mathbf{T}$ which consists entirely of degenerate J_x's. In fact, suppose there were such an interval I, and let $0 < \varepsilon <$ length of I. Then there is a positive integer k such that for any $z \in M$, the set $\{z, f(z), \cdots, f^k(z)\}$ is ε-dense in M. (This is true for any minimal set; the proof is straightforward.) Thus, if $z \in M$, there is a j, $0 \leqq j \leqq k$, such that $f^j(z) \in I$, so $J_{f^j(z)} = \{f^j(z)\}$, and hence $J_{f^k(z)} = \{f^k(z)\}$. Now, since $f^k(M) = M$, we have $J_x = \{x\}$, for all $x \in M$. M consists precisely of the endpoints of the intervals J_y $(y \in \mathbf{T})$, so $J_y = \{y\}$ for all $y \in \mathbf{T}$. It follows from (vi) of Theorem 1 that f is in fact a homeomorphism, and we have a contradiction.

Now we can show that all continuous maps of the circle without periodic points arise from homeomorphisms as indicated at the beginning of the paper. If f restricted to the minimal set M is homeomorphism (in which case $J_{f(x)}$ is non-degenerate whenever J_x is), we simply modify f to be a homeomorphism on the interior of the non-degenerate J_x's. Otherwise, proceed as follows. Define an equivalence relation $\sim$ on $\mathbf{T}$ by $x_1 \sim x_2$ if $J_{x_1} = J_{x_2}$, and let $\mathbf{T}'$ be the quotient space $\mathbf{T}/\sim$. It is easy to see that $\mathbf{T}'$ is itself homeomorphic to the circle (recall that at most countably many equivalence classes are non-trivial). Moreover, by (iv) and (ix) of Theorem 1, f induces a homeomorphism f' of $\mathbf{T}'$. Now, if $x \in \mathbf{T}$, and y an endpoint of J_x, then $y \in M$ and $y \sim x$. Thus if $\pi : \mathbf{T} \to \mathbf{T}'$ is the canonical map, $\mathbf{T}' = \pi(M)$, and $\mathbf{T}'$ is minimal under f'. (If follows that f is semi-conjugate to a rotation of the circle.)

Let $L = [x \in \mathbf{T} \mid J_x$ is non-degenerate$]$, $A = [x \in L \mid J_{f(x)} = \{f(x)\}]$ (by assumption $A \neq \emptyset$) and let B be a maximal subset of L such that $A \subset B$, $x' \in B$ whenever $x \in B$ and $J_x = J_{x'}$, and if $x_1, x_2 \in B$, then $f^m(J_{x_1}) \neq J_{x_2}$ $(m \geqq 0)$. (That is, $B \supset A$ and consists of one J_x from each "orbit".) Let $B' = \pi(B)$; B' is a (finite or) countable subset of $\mathbf{T}'$. $B' = \{x'_1, x'_2, \cdots\}$ and the orbits $\mathcal{O}(x'_k)$ are pairwise disjoint.

Next we obtain, from the $\mathcal{O}(x'_k)$, a non-minimal homeomorphism of the circle $\mathbf{T}^*$ in the usual fashion. To be precise, to each $f^j(x'_k)$ $(j = 0, \pm 1, \pm 2, \cdots;$ $h = 1, 2, \cdots)$ associate an open interval $I_j^{(k)}$ on $\mathbf{T}^*$ so that $I_j^{(k)}$ appears in the same cyclic order as the points $f^j(x'_k)$ (say by a diagonal process) and such that the sum of the lengths of the $I_j^{(k)}$ is one. If $I^* = \bigcup_{-\infty<j<\infty\, k\geqq 1} I_j^{(k)}$, define $\varphi : I^* \to I^*$ so that φ maps $I_j^{(k)}$ linearly onto $I_{j+1}^{(k)}$. Clearly φ and φ^{-1} are uniformly continuous, so φ extends to a homeomorphism, still called φ, of the circle $\mathbf{T}^*$ ($=$ closure I^*). Finally, we identify z_1 and z_2 of $\mathbf{T}^*$ if z_1 and z_2 are in the same $I_j^{(k)}$, where $j \geqq 0$, and $x'_k \in \pi(A)$. The minimal set on the quotient space is clearly isomorphic with M, and modifying the induced map on the interior of the non-degenerate intervals, we can recover f.

By the above discussion, f has a homomorphic image which is a minimal homeomorphism of $\mathbf{T}$, and f is itself (essentially) a homomorphic image of a homeomorphism of $\mathbf{T}$ without periodic points. Since all such homeomorphisms are uniquely ergodic ([1]), it follows that f is uniquely ergodic.

Reference

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